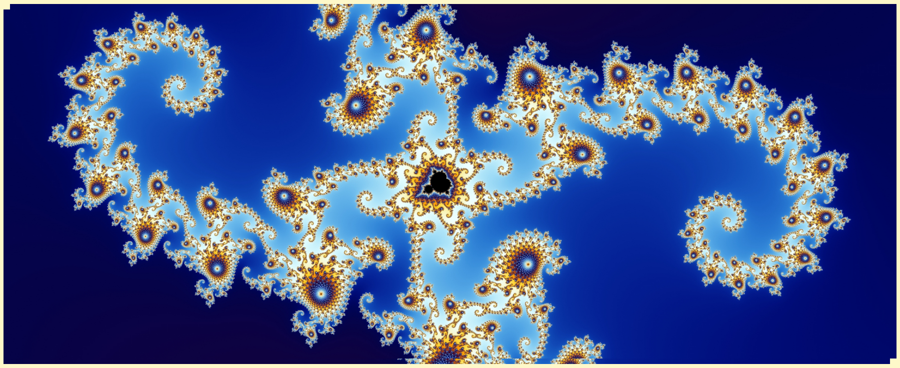
Math 201: Linear Algebra
Week 14

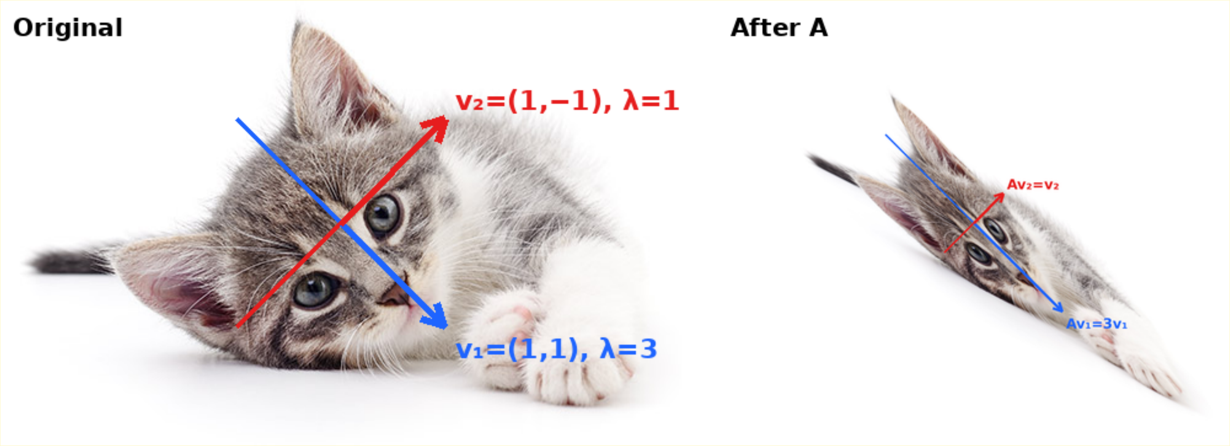
Today:

1. Section 7.3: Finding Eigenvectors
2. Section 7.4: Matrix Diagonalization
3. Review of complex numbers (If we have time)



$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \longrightarrow$$

Note: $A\vec{v} = \lambda\vec{v} \iff \vec{v} \in \ker(A - \lambda I)$. Find <u>all</u> the eigenvectors of $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.



Define:

1. Eigenspace
2. Geometric multiplicity
3. Eigenbasis

Example: $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Find: all eigenspaces, and an eigenbasis.

Diagonalization: Suppose $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an eigen basis. Let $S = [\vec{v}_1 \cdots \vec{v}_n]$.

Then $AS = [A\vec{v}_1 \cdots A\vec{v}_n] = [\lambda_1\vec{v}_1 \cdots \lambda_n\vec{v}_n] = SD$

Motivation:

1. Compute A^t & find $\lim_{t\to\infty} A^t$ (I.e. for dynamical systems)

2. Decouple PDES: $u(x,t)$, $v(x,t)$, $u_x = \frac{\partial u}{\partial x}$, $u_t = \frac{\partial u}{\partial t}$ etc:

- $u_t = 3u_{xx} + v_{xx}$
- $v_t = 2v_{xx}$

$\Rightarrow \vec{u}_t = \begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} u_{xx} \\ v_{xx} \end{bmatrix}}_{= \vec{u}_{xx}}$

Let $S = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

Then $A\vec{u} = (SDS^{-1})\vec{u}$. Let $\vec{w} = S^{-1}\vec{u}$.

$$\vec{u}_t = A\vec{u}_{xx} = SDS^{-1}\vec{u}_{xx}$$

$$\Rightarrow S^{-1}\vec{u}_t = DS^{-1}\vec{u}_{xx}$$

$$\Rightarrow \vec{w}_t = D\vec{w}_{xx}$$

$$\Rightarrow w_{1,t} = 3w_{1,xx}$$

$$w_{2,t} = 2w_{2,xx}$$

Scalar Heat Equation

$$u_t = u_{xx} \qquad x \in [0,1], \quad u(0,t) = u(1,t) = 0.$$
$$u(x,0) = f(x)$$

- Operator: $L = \partial_{xx} : C^\infty \to C^\infty$.
- Diagonalize: Find $\phi(x)$ s.t. $L\phi = \lambda\phi$. that is, such that

$$\phi''(x) = \lambda\phi(x), \quad \phi(0) = \phi(1) = 0$$

Answer: $\phi_n(x) = \sin(n\pi x) \qquad \lambda_n = -n^2\pi^2$

- Eigen basis (Fourier series)

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) \sin(n\pi x)$$

- Substitute into $u_t = u_{xx}$:

$$\sum_{n=1}^{\infty} c_n'(t) \sin(n\pi x) = \sum_{n=1}^{\infty} c_n(t) \left(-n^2\pi^2 \sin(n\pi x)\right)$$

$$\Rightarrow \qquad c_n'(t) = -n^2\pi^2 c_n(t)$$

$\Rightarrow$ ∞-many ODEs. One per $n \in \mathbb{Z}$.

- Solve: $c_n(t) = c_n(0) e^{-n^2\pi^2 t}$

At $t=0$, $u(x,0) = f(x)$

$$\Rightarrow \quad c_n(0) = 2\int_0^1 f(x) \sin(n\pi x)\, dx$$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} c_n(0) e^{-n^2\pi^2 t} \sin(n\pi x).$$

When is a matrix diagonalizable?

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$ compute geometric and algebraic multiplicities.